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## On periodic P.I. rings and locally finite rings

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## ON PERIODIC P. I. RINGS AND LOCALLY FINITE RINGS

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An element  $x$  of a ring  $R$  is called *periodic* if there exist distinct positive integers  $m, n$  for which  $x^m = x^n$ . Especially,  $x$  is called *potent* if  $x^m = x$  for some positive integer  $m > 1$ . A ring  $R$  is called *periodic* if all elements of  $R$  are periodic. It is easily seen that a periodic ring  $R$  has the property that every element of  $R$  is expressible as a sum of a potent element and a nilpotent element. However it is not known whether a ring  $R$  with this property is periodic or not. On the other hand, by a result of the author and H. Tominaga [6], if  $R$  is a P. I. ring in which every element is the sum of two idempotents, then  $R$  is periodic. In this paper, we shall prove that a P. I. ring  $R$  in which every element is expressible as a sum of two periodic elements, is periodic.

We shall next consider the local finiteness of a periodic P. I. ring. A ring  $R$  is said to be *locally finite* if any finitely generated subring of  $R$  is a finite ring. Let  $R$  be a periodic P. I. ring, and  $S$  a finitely generated subring of  $R$ . We shall show that the additive group of  $S$  is finitely generated and that some power of  $S$  is a finite ring. Consequently a P. I. ring  $R$  is locally finite if and only if  $R$  is periodic and the additive group of  $R$  is a torsion group. Using this, we shall give a characterization of a locally finite ring.

We begin with the following lemma.

**Lemma 1.** *Let  $R$  be a ring. Then  $R$  is periodic if and only if all prime factor rings of  $R$  are periodic.*

*Proof.* Suppose that all prime factor rings of  $R$  are periodic. For each  $x \in R$ , let  $S(x) = \{x^n - x^{n+1}f(x) \mid n > 0 \text{ is an integer, } f(t) \in \mathbf{Z}[t]\}$ , which is multiplicatively closed. By virtue of [3, Proposition 2],  $R$  is periodic if and only if  $0 \in S(x)$  for all  $x \in R$ . Assume, to the contrary, that there exists  $a \in R$  such that  $0 \notin S(a)$ . Then, by Zorn's lemma, we can find an ideal  $I$  of  $R$  which is maximal with respect to the property that  $S(a) \cap I = \emptyset$ . It is easy to check that  $I$  is a prime ideal of  $R$ . Hence  $R/I$  is periodic by hypothesis. But this contradicts the fact that  $S(a) \cap I = \emptyset$ .

A ring  $R$  is said to be *of bounded index (of nilpotence)* if there is a positive integer  $n$  such that  $a^n = 0$  for any nilpotent element  $a$  in  $R$ . The least

such integer is called the *index* of  $R$ . We shall show that a periodic ring of bounded index is a P. I. ring. Let  $G$  denote the symmetric group of degree  $n$ . The identity

$$s_n = \sum_{\sigma \in G} \text{sgn}(\sigma) X_{1\sigma} X_{2\sigma} \cdots X_{n\sigma}$$

is called the *standard identity of degree  $n$* .

**Proposition 1.** *Let  $n$  be a positive integer and let  $R$  be a periodic ring of index  $n$ . Then  $R$  satisfies the polynomial identity  $(s_{2n})^n$ .*

*Proof.* Let  $J$  denote the Jacobson radical of  $R$ , and  $x$  an element of  $J$ . Then there exist positive integers  $p, q$  such that  $x^{p+q} = x^p$ . By [4, Theorem 1.2.3] all elements of  $J$  are right-quasi-regular. Hence there exists  $y \in R$  such that  $(-x^q) + y + (-x^q)y = 0$ . Then  $x^p = x^p + x^p(-x^q + y - x^q y) = (x^p - x^{p+q}) + (x^p - x^{p+q})y = 0$ . This implies that  $J$  is a nil ideal. Let  $P$  be a primitive ideal of  $R$ . By [7, Theorem 2.3]  $R/P = M_t(D)$  for some division ring  $D$  and some positive integer  $t \leq n$ . Since  $D$  is a periodic division ring,  $D$  is commutative by [4, Lemma 3.1.3]. Hence  $R/P$  satisfies the standard identity  $s_{2n}$  of degree  $2n$  by [8, Theorem 1.4.1]. Since  $R/J$  can be embedded in the direct product of all primitive factor rings of  $R$ ,  $R/J$  also satisfies the identity  $s_{2n}$ , in other words,  $s_{2n}(a_1, a_2, \dots, a_{2n}) \in J$  for all elements  $a_1, a_2, \dots, a_{2n}$  in  $R$ . Since  $J$  is a nil ideal of index at most  $n$ , we have that  $s_{2n}(a_1, a_2, \dots, a_{2n})^n = 0$  for all  $a_1, a_2, \dots, a_{2n} \in R$ . This completes the proof.

If  $R$  is a periodic ring, each element  $x$  in  $R$  can be expressed in the form  $y + w$ , where  $y^n = y$  for some  $n = n(y) > 1$  and  $w$  is nilpotent (e.g., see [2, Lemma 1]). However it is not known whether this property characterizes a periodic ring. On the other hand, by [6, Theorem 2], if  $R$  is a P. I. ring in which every element is the sum of two idempotents then, for any  $x \in R$ ,  $x^3 - x$  is nilpotent. Hence  $R$  is periodic by [3, Proposition 2]. We shall now prove the following

**Theorem 1.** *Let  $R$  be a P. I. ring. If every element of  $R$  is expressed as a sum of two periodic elements, then  $R$  is periodic.*

*Proof.* By virtue of Lemma 1, we may assume that  $R$  is a prime ring. Then, by [5, Theorem 1.4.2] the center  $C$  of  $R$  is nonzero. We claim that  $C$  is periodic. Let  $c$  be a nonzero element of  $C$ . Then, by hypothesis, there

exist  $x, y \in R$  such that  $c = x + y$ ,  $x^m = x^n$  for some  $m > n > 0$ , and  $y^p = y^q$  for some  $p > q > 0$ . Then  $(c - y)^m = (c - y)^n$ , and so  $c^m - c^n = zy$  for some  $z \in C[y] (\subset R)$ . If  $c^m - c^n$  is nilpotent, then  $c^m = c^n$ , because  $C$  is an integral domain. Assume now that  $c^m - c^n$  is not nilpotent. Then  $e = y^{(p-q)q}$  is a nonzero idempotent and  $y^q e = ey^q = y^q$ . Therefore we have that  $(c^m - c^n)^q (ae - a) = 0$  for all  $a \in R$ . Let us put  $L = \{ae - a \mid a \in R\}$ . Then  $L$  is a left ideal of  $R$ , and as seen above,  $(c^m - c^n)^q L = 0$ . Since  $(c^m - c^n)^q \neq 0$  and since  $R$  is a prime ring, we obtain  $L = 0$ , that is,  $e$  is a right identity of  $R$ . We can similarly prove that  $e$  is a left identity of  $R$ . Hence  $e$  is the identity of  $R$ . We shall now prove that the characteristic of  $R$  is nonzero. Assume, to the contrary, that the characteristic of  $R$  is zero. Then we may assume that  $R$  contains the ring  $\mathbf{Z}$  of integers as a subring. By hypothesis, there exist two periodic elements  $v, w \in R$  such that  $3 = v + w$ . Obviously the subring  $S = \mathbf{Z}[v, w]$  of  $R$  generated by  $v$  and  $w$  over  $\mathbf{Z}$  is a commutative ring which is integral over  $\mathbf{Z}$ . By [1, Theorem 5.10] there exists a prime ideal  $P$  of  $S$  such that  $P \cap \mathbf{Z} = 0$ . Consider now the factor ring  $\bar{S} = S/P$ . Then  $\bar{S}$  is an integral domain which is integral over  $\mathbf{Z}$ . So, without loss of generality, we may assume that  $\bar{S}$  is a subring of the field  $C$  of complex numbers. In general, if  $a$  is a periodic element of  $C$ , then the absolute value  $|a|$  of  $a$  is either 0 or 1. Hence we have  $3 = |\bar{v} + \bar{w}| \leq |\bar{v}| + |\bar{w}| \leq 2$ , which is a contradiction. Therefore the characteristic of  $R$  is nonzero. Let  $F$  denote the prime field of  $C$ . Since  $x$  and  $y$  are integral over  $F$ ,  $c = x + y$  is integral over  $F$ . Hence  $c$  generates a finite subring of  $C$ , and so  $c$  is periodic. Therefore we proved that  $C$  is a periodic field. By [8, Corollary 1.6.28],  $R$  is a simple P. I. ring. Hence, by Kaplansky's theorem [8, Theorem 1.5.16],  $R$  can be identified with the matrix ring  $M_t(D)$  over a division ring  $D$  which is finite dimensional over  $C$ . Then  $D$  is also periodic, and hence  $D$  is commutative. Thus we get  $C = D$ . Therefore  $R = M_t(C)$  is periodic.

We shall next consider the finitely generated subrings of a periodic P. I. ring. Clearly a periodic P. I. ring need not be locally finite. For example, the subring

$$\begin{pmatrix} 0 & \mathbf{Z} \\ 0 & 0 \end{pmatrix} \text{ of } M_2(\mathbf{Z})$$

is a finitely generated periodic commutative ring, but this is not a finite ring. We shall prove the following:

**Theorem 2.** *Let  $R$  be a periodic P. I. ring and let  $S$  be a finitely gener-*

ated subring of  $R$ . Then the additive group  $S^+$  of  $S$  is a finitely generated abelian group. Moreover there exists a positive integer  $n$  such that  $S^n$  is a finite ring. In particular, if  $S$  has an identity, then  $S$  is finite.

*Proof.* Let  $t(S)$  denote the torsion submodule of the  $\mathbf{Z}$ -module  $S$ . Then  $t(S)$  is an ideal of  $S$  and  $S/t(S)$  is torsion-free. Let  $x$  be an element of  $S/t(S)$ . Then  $x^{m+n} = x^m$  for some positive integers  $m, n$ . Then we can easily see that  $x^{m+n}$  is an idempotent. Since  $(2x^{m+n})^{p+q} = (2x^{m+n})^p$  for some positive integers  $p$  and  $q$ , we obtain a positive integer  $h$  such that  $hx^{m+n} = 0$ . Since  $S/t(S)$  is torsion-free, we conclude that  $x^{m+n} = 0$ . Thus  $S/t(S)$  is a nil ring. Since  $S/t(S)$  is also a finitely generated P. I. ring, there exists a positive integer  $n$  such that  $(S/t(S))^n = 0$  by [8, Proposition 1.6.34]. Hence we have  $S^n \subset t(S)$ . Let  $c_1, c_2, \dots, c_m$  generate the subring  $S$ . Then  $A = \{c_{i_1}c_{i_2} \cdots c_{i_n} \mid 1 \leq i_j \leq m\}$  is a finite set, and hence there exists a positive integer  $k$  such that  $kA = 0$ . Hence we have  $kS^n = 0$ . Let  $B$  denote the set  $\{c_{i_1}c_{i_2} \cdots c_{i_p} \mid 1 \leq i_j \leq m, 1 \leq p \leq n\}$ . Then we can easily see that

$$kS = \sum_{b \in B} \mathbf{Z}kb.$$

Hence  $kS$  is a finitely generated  $\mathbf{Z}$ -module. Let  $S'$  denote the ring  $S/kS$  and let us write  $k = \prod_{i=1}^t p_i^{k_i}$  where the  $p_i$  are distinct primes and  $k_i > 0$  for all  $i$ . Then, for each  $i$ ,  $S'_i = \{a \in S' \mid p_i^{k_i}a = 0\}$  is a subring of  $S'$  and  $S'$  is the direct sum of  $S'_1, S'_2, \dots, S'_t$ . We shall show that  $S'$  is finite. To show it, it suffices to prove that  $S'_i$  is finite for each  $i = 1, 2, \dots, t$ . Hence, without loss of generality, we may assume that  $k = p^h$  for some prime  $p$  and some positive integer  $h$ . Let us set  $I = pS'$ . Then  $I^h = 0$  and  $p^{h-1}I = 0$ . Then the ring  $S'/I$  is a finitely generated periodic algebra over  $\mathbf{Z}/p\mathbf{Z}$  satisfying a polynomial identity. Hence  $S'/I$  is a finite dimensional algebra over  $\mathbf{Z}/p\mathbf{Z}$  by [4, Theorem 6.4.3]. Let  $S'/I = \{a_0 + I, a_1 + I, \dots, a_d + I\}$  where  $a_0 = 0, a_1, \dots, a_d$  are elements of  $S'$ . Then we can choose elements  $b_1, b_2, \dots, b_f$  of  $I$  such that  $a_1, a_2, \dots, a_d, b_1, b_2, \dots, b_f$  generate  $S'$ . For any  $i, j$  with  $1 \leq i, j \leq d$ , we have a unique integer  $t(i, j)$  with  $1 \leq t(i, j) \leq d$  such that  $a_i a_j \equiv a_{t(i, j)}$  modulo  $I$ . Similarly we have a unique integer  $s(i, j)$  such that  $a_i + a_j \equiv a_{s(i, j)}$  modulo  $I$ . Let us now set  $x_{ij} = a_i a_j - a_{t(i, j)}$  and  $y_{ij} = a_i + a_j - a_{s(i, j)}$  for each  $1 \leq i, j \leq d$ . Let  $J$  denote the subring of  $S'$  generated by  $x_{\alpha\beta}, y_{\mu\nu}, b_\lambda, a_\gamma x_{\alpha\beta}, a_\gamma y_{\mu\nu}, a_\gamma b_\lambda, x_{\alpha\beta} a_\gamma, y_{\mu\nu} a_\gamma, b_\lambda a_\gamma$  for  $1 \leq \alpha, \beta, \gamma \leq d, 1 \leq \mu, \nu \leq d$ , and  $1 \leq \lambda \leq f$ . Then  $J$  is a finitely generated subring of  $I$ . Since  $I^h = 0$  and  $p^{h-1}I = 0$ ,  $J$  must be finite. We can now easily see that each element  $x$  of  $S'$  can be uniquely expressed in the form  $a_i + z$ , where  $0 \leq i \leq d$  and  $z \in J$ . This implies that  $I = J$ . Therefore  $S'$  is a finite ring. Conse-

quently  $S$  is a finitely generated  $\mathbf{Z}$ -module. Since the additive group of  $S^n$  is a torsion group,  $S^n$  is a finite ring. In particular, if  $S$  has an identity, then  $S^n = S$ , and hence  $S$  is finite.

As an immediate consequence of this theorem, we obtain the following :

**Corollary 1.** *Let  $R$  be a P. I. ring. Then  $R$  is locally finite if and only if  $R$  is periodic and the additive group of  $R$  is a torsion group.*

A ring  $R$  is said to be of *locally bounded index* if every finitely generated subring of  $R$  is of bounded index. Combining Corollary 1 with Proposition 1, we obtain the following characterization of a locally finite ring.

**Corollary 2.** *A ring  $R$  is locally finite if and only if  $R$  is a periodic ring of locally bounded index and the additive group of  $R$  is a torsion group.*

The following example due to Golod and Shafarevitch shows that a finitely generated periodic ring with torsion additive group need not be finite.

**Example 1.** Let  $p$  be a prime number. By [4, Theorem 8.1.3], there exists an infinite dimensional nil algebra  $A$  over  $\mathbf{Z}/p\mathbf{Z}$  generated by three elements. Clearly  $A$  is generated by those three elements as a ring. Note that those elements generate infinite subsemigroup of the multiplicative semigroup of  $R$ .

As another corollary of Theorem 2, we obtain the following

**Corollary 3.** *Let  $R$  be a P. I. ring. Then the following statements are equivalent :*

- (1)  $R$  is periodic.
- (2) For any finitely generated subring  $S$  of  $R$ , there exists a positive integer  $n$  such that  $S^n$  is a finite subring.
- (3) For any finitely generated subring  $S$  of  $R$ , there exists a finite ideal  $I$  of  $S$  such that  $S/I$  is a nilpotent ring.
- (4) The ideal  $t(R) = \{a \in R \mid na = 0 \text{ for some positive integer } n\}$  is locally finite and  $R/t(R)$  is a nil ring.

*Proof.* The implication (1)  $\Rightarrow$  (2) follows from Theorem 2 and (2)  $\Rightarrow$  (3) is obvious.

(3)  $\Rightarrow$  (1). Let  $x$  be an element of  $R$ , and  $S$  denote the subring of  $R$

generated by  $x$ . Then there exists a finite ideal  $I$  of  $S$  such that  $S/I$  is nilpotent. This implies that some power of  $x$  generates a finite subring. Hence there exist distinct positive integers  $m, n$  such that  $x^m = x^n$ .

(1)  $\Leftrightarrow$  (4). Assume that  $R$  is periodic. By Corollary 1  $t(R)$  is locally finite. We also know that  $R/t(R)$  is a nil ring by the proof of Theorem 2.

Conversely, suppose that (4) holds, and let  $x$  be an element of  $R$ . Then some power of  $x$  generates a finite subring of  $R$ , and hence  $x$  is periodic.

A ring  $R$  is periodic if and only if each subsemigroup of  $R$  generated by a single element is finite. If  $R$  is a commutative periodic ring, then all finitely generated subsemigroups of  $R$  are finite. However Example 1 shows that this does not remain valid for noncommutative periodic rings. Thus we have the following

**Conjecture.** Let  $R$  be a periodic P. I. ring. Then all finitely generated subsemigroups of  $R$  are finite.

#### REFERENCES

- [ 1 ] M. F. ATIYAH and I. G. MACDONALD : Introduction to Commutative Algebra, Addison-Wesley, Reading, Mass. 1969.
- [ 2 ] H. E. BELL : A commutativity study for periodic rings, Pacific J. Math. 70 (1977), 29–36.
- [ 3 ] M. CHACRON : On a theorem of Herstein, Canad. J. Math. 21 (1969), 1348–1353.
- [ 4 ] I. N. HERSTEIN : Noncommutative Rings, Math. Assoc. America, Carus Monograph No. 15, 1968.
- [ 5 ] I. N. HERSTEIN : Rings with Involution, Univ. of Chicago, Chicago, 1976.
- [ 6 ] Y. HIRANO and H. TOMINAGA : Rings in which every element is the sum of two idempotents, Bull. Austral. Math. Soc. 37 (1988), 161–164.
- [ 7 ] I. KAPLANSKY : Topological representations of algebras, II, Trans. Amer. Math. Soc. 68 (1950), 62–75.
- [ 8 ] L. H. ROWEN : Polynomial Identities in Ring Theory, Academic Press, 1980.

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